I.T.I.S. "E. DIVINI" SAN SEVERINO MARCHE

The number " *e* "

Tuesday 10th January 2006

The number " e "

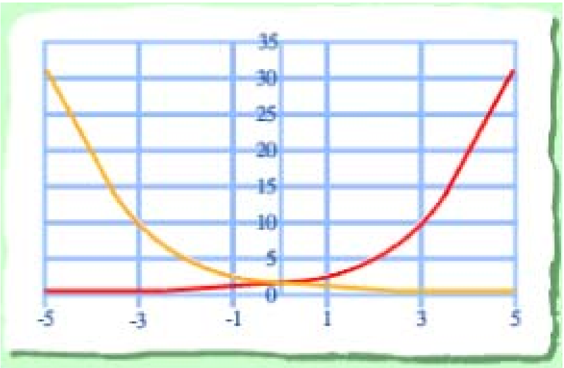
French children are told a story in which you imagine having a pond with water lily leaves floating on the surface. The lily doubles in size every day and if left unchecked will smother the pond in 30 days, killing all the other living things in the water. Day after day the plant seems small and so you decide to leave it grow until it half-covers the pond, before cutting it back. On what day will that occur? The 29th day, and then you will have just one day to save the pond

There is an old tale of a boy who did a great service for a King. The King offered the boy any prize that he wanted, so the boy asked for a quantity of rice. "Put one grain of rice on the first square of a chessboard" said the boy, "and put two on the second square, then double to 4 for the third square and keep doubling until you reach the last square of the board". The King felt relieved that the boy had asked for such a modest prize.

What the King didn't realise is that on the last square alone there would be 9.2×10^{18} grains, which allowing for a million grains per sack is roughly 10 trillion sacks.

What the King needed was a mathematical model for functions that grow rapidly. This is an example of an exponential function. The number of grains on the 5th square is $2^{5-1} = 2^4$ On the *n*th square it is 2^{n-1}. A function with an exponent.

Exponential functions are very good for modelling situations with rapid growth or rapid decay, since the exponent itself can be positive or negative.



The spread of bacteria can be modelled by an exponential function, since the quantity of bacteria will double in a relatively constant time.

The same has been true of human world population growth.

A good example of exponential decay is radioactive material. This will diminish in size by a fixed factor in equal time intervals.

But the general principle behind exponential growth is that the larger a number gets, the faster it grows. Any exponentially growing number will eventually grow larger than any other number which grows at only a constant rate for the same amount of time (and will also grow larger than any function which grows only *subexponentially*). This is demonstrated by the classic riddle in which a child is offered two choices for an increasing weekly allowance: the first option begins at 1 cent and doubles each week, while the second option begins at $1 and increases by $1 each week. Although the second option, growing at a constant rate of $1/week, pays more in the short run, the first option eventually grows much larger:

Questions

1. A radioactive substance reduces in mass. Initially it had a mass of 20g. After 5 days its' mass had reduced to 14.5g. Assuming exponential decay suggest a function that would model this data.
2. A savings account pays an annual compound interest of 6%. By what factor does the account grow every year ?
3. The number of people who have seen a notice on the sixth form notice board is modelled by the function: $P=N(1 - 2^{-t})$. t is the time elapsed in days and N is constant. If 150 people have seen it in 2 days. How many people are in the sixth form ?
4. The number of bacteria in a colony is modelled by the function: $B = 300 \times K^t$
   t is the time elapsed in hours and K is constant. If there are 1200 bacteria at the end of the second hour, how many are there at the end of the third hour ?
5. The amount of a certain drug (in mg) reaching a patients bloodstream t seconds after injection is modelled by the formula: $B = 4 \times 2.2^{-0.3t}$ Find the amount of the drug in the patients bloodstream 5 seconds after injection.
6. The annual profit made by a company on a product which has been sold for y years is modelled by the function:
   $$P = 15000 - 8000\left(\frac{1}{3}\right)^y$$
   Find the profit made in the third year

A special case

There is one particular base for logarithms that is given special importance. It is base e. The number *e* is 2.718 281 828 4 (to ten decimal places). It was first named *e* by the Swiss mathematician Leonhard Euler (1707-1783).

It is defined as $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

n	e
10	2.59374246
100	2.704813829
1000	2.716923932
10000	2.718145927
100000	2.718268237
1000000	2.718280469
10000000	2.718281694
100000000	2.718281786
1000000000	2.718282031
10000000000	2.718282053

Which means that as n gets bigger and bigger the value of this function gets closer and closer to e.
Test it out with a large n.

Work out the value of $\left(1 + \frac{1}{100}\right)^{100}$

You should get 2.705 (to 3 d.p.).
That's quite close to **e**
Now try it out with n=1000.

e is an irrational number. More than that, like π it is a transcendental number, which means that it cannot be found as the solution of polynomial equation with integer coefficients.
So you could not solve a quadratic and get the answer *e*.

It turns out that many natural phenomena can be modelled by exponential functions using e as their base. The exponential function e^x has a very important property, which makes it particularly special amongst exponential functions.

The function $f(x) = e^x$ is an exponential function with base *e*.
Where *e* is a fundamental constant whose value is approximately 2.718.

.

Examples of exponential growth

- Biology.
  - Microorganisms in a culture dish will grow exponentially, at first, after the first microorganism appears (but then logistically until the available food is exhausted, when growth stops).
  - A virus (SARS, West Nile, smallpox) of sufficient infectivity ($k > 0$) will spread exponentially at first, if no artificial immunization is available. Each infected person can infect multiple new people.
  - Human population, if the number of births and deaths per person per year were to remain constant (but also see logistic growth).
  - Many responses of living beings to stimuli, including human perception, are logarithmic responses, which are the inverse of exponential responses; the loudness and frequency of sound are perceived logarithmically, even with very faint stimulus, within the limits of perception. This is the reason that exponentially increasing the brightness of visual stimuli is perceived by humans as a smooth (linear) increase, rather than an exponential increase. This has survival value. Generally it is important for the organisms to respond to stimuli in a wide range of levels, from very low levels, to very high levels, while the accuracy of the estimation of differences at high levels of stimulus is much less important for survival.
- Electroengineering
  - Charging and discharging of capacitors and changes in current in inductors are also exponential growth and decay phenomena. Engineers use a rule of five time constants to estimate when a steady state has been reached.
- Computer technology
  - Processing power of computers. See also Moore's law and technological singularity (under exponential growth, there are no such singularities).
  - Internet traffic growth.
- Investment. The effect of compound interest over many years has a substantial effect on savings and a person's ability to retire. See also rule of 72
- Physics
  - Atmospheric pressure decreases exponentially with increasing height above sea level, at a rate of about 12% per 1000m.
  - Nuclear chain reaction (the concept behind nuclear weapons). Each uranium nucleus that undergoes fission produces multiple neutrons, each of which can be absorbed by adjacent uranium atoms, causing them to fission in turn. If the probability of neutron absorption exceeds the probability of neutron escape (a function of the shape and mass of the uranium), $k > 0$ and so the production rate of neutrons and induced uranium fissions increases exponentially, in an uncontrolled reaction.
  - Newton's law of cooling $T = A + De^{-kt}$ where T is temperature, t is time, and, A, D, and $k > 0$ are constants, is an example of exponential decay.